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Mathematical Analysis of the curve assumed by LED string lights when in static equilibrium

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Abstract -Beautiful LED String lights are used around the world during festivals, celebrations, get together etc. However, its aesthetic nature comes with critical hazards. To avoid such dangers and minimize expenses, this essay explores the topic “Mathematical Analysis of the curve assumed by LED string lights when in static equilibrium” using Calculus of Variations, specifically Euler Lagrange Equation and Beltrami Identity.

This paper explores the subject step wise with consideration of certain assumptions, however most of the paramount factors are included. It investigates the shape of the curve assumed by LED string lights: catenary or parabola? The result of part 1 is applied in part 2 investigation, which makes use of calculus of variations and investigates the minimum distance between the two end points of the hung LED lights such that it touches the ground. The third part employs graphs and tables to maximize the area covered by the lights, thereby minimizing resources spent by consumers.The essay concludes that LED String Lights hung on the wall assume the shape of a catenary with the equation C*cosh(x/c). With this result it further deduces the optimum distance between the two end points of the hung lights which maximizes the area covered and maintains a safe distance above the ground.

Index Terms— application of catenary, catenary, LED String Lights, safety with LED String Lights

1 INTRODUCTION

Sometimes the simplest things in life become very complex for us. Although immediately one may not be able to relate it with the field of Mathematics, it is a matter of fact that almost everything in our lives and natural surroundings are related to Mathematics. When I started studying in IB, I became more conscious and vigilant of my surroundings and the activities going on in it. One of the many remarkable notes I once made exposed me to a new field of Mathematics: Calculus of Variations.

During my Christmas vacation; when I was decorating the house with a fixed length of LED String lights I faced the following challenges:

1. How should I decorate the wall area with limited length LED strings lights so as to cover maximum area?
2. At which height should I hang these LED lights so that it does not cause harm to others?
3. What distance should I maintain between two ends while hanging the LED lights so as to decorate the whole house with minimum length of LED lights?

When I tried to solve these problems, it took me around two days with several trials and errors while hanging the lights. I was in a big dilemma of saving costs as well beautifying the whole house. To solve this dilemma, I began to look at this problem with a Mathematical perspective. Since I love Mathematics, this dilemma motivated me to research on this topic and hence I decided to use Calculus of Variations to solve the problem.

Since maximization of area can only be done if the equation of the curve is known, I aim to mathematically model the hanging LED Strip lights, deduce the equation of the curve and calculate the distance between the two points on the wall for which the string lights just touches the ground. With this result I can adjust the distance between the two points on the wall and analyze the importance of changing the significance of the curve mathematically. Along with this I also aim to calculate the distance between the endpoints of the curve for which it covers maximum area.

2 MATHEMATICAL ANALYSIS

2.1 Part 1: Deriving the Equation of the curve:

In my Mathematical Analysis, I will be deriving the equation of the curve using Calculus of Variations and categorizing it depending upon the result. For the calculation made further in the investigation, it is assumed that the weight of the LED lights is negligible compared to that of the whole cable of LED lights and that the only external force acting on the cable is the force of gravitation. The variables known in this case are; the height of the wall is 10ft (3.08 meter) and the length of the String of Lights is 72ft (22 meter). The shape of the curve formed by the hanging LED string lights is somewhat that of a parabola or Catenary. The difficulty is in distinguishing between a parabola and a catenary because of the minute difference between them.

- **Difference between catenary and parabola**

Mathematically and scientifically there is a major difference between a catenary and a parabola.

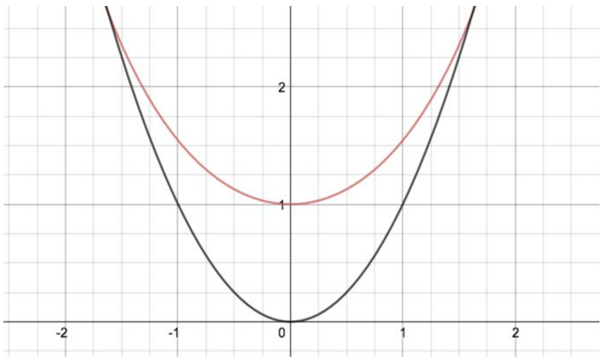


Figure 1: Red line: Catenary andBlack

Figure 2: Red Line: Catenary and Black line: Parabola

If the LED String lights would be supporting a deck or a heavy group of lights at its base such that the weight of the cable is negligible compared to the weight of its supporting material, it would take the shape of a parabola. However, in this case, the LED lights are spread out uniformly and it is also known that the weight of an LED bulb is negligible therefore there is a fair chance that it is assuming the shape of a catenary.

Why do I think that the curve assumed by the LED string light is a parabola?

Before deriving the equation of the curve assumed by the LED string lights
Let us examine the probability of the lights assuming which curve parabola or catenary?

If the lights are supporting any weight (which they are not), then every small length of the light will have the same vertical component of tension force per foot of the cable, Hence the second derivative $f''(x)$ of the curve will be constant equal to c.

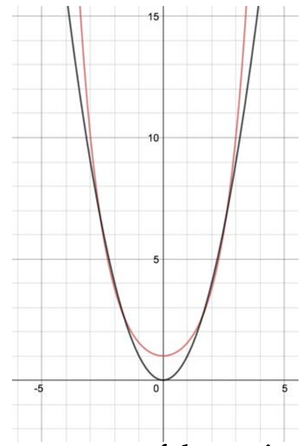
The general equation of a parabola is

$$\boldsymbol{f(x) = ax^2 + bx + c}$$

$$\boldsymbol{f'(x) = 2ax + b}$$

$$\boldsymbol{f''(x) = 2a}$$

Hence the shape of the curve assumed by the string lights would be a parabola if and only if it supported a heavy material or group of lights, whose weight is much greater than that of the LED string light cable.



The definition of a catenary curve is: "The curve a hanging flexible wire or chain assumes when supported at its ends and acted upon by a uniform gravitational force."
Examining the real life example presented in this exploration, the LED string lights are supported by its ends and only support its own weight. It is also acted upon by gravity.

Following Diagram illustrates the horizontal and vertical component of the tension force acting on infinitesimal lengths of the curve and its relation with the curve's slope:

[1]

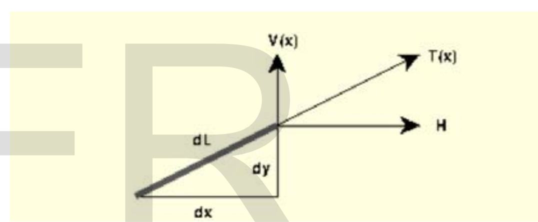


Figure 3

In general, the horizontal component of tension is constant because if there would be net force, the chain or rail or bridge would accelerate. However, the vertical component of tension force (which is f′ (x)) changes because it resists the gravitational force. The change in the vertical component of force is $f''(x)$.
Hence for a cable supporting its own weight, it is essential to calculate the weight per unit of the cable (which can be calculated using the Pythagoras theorem). The change in the vertical component of tension force ($f''(x)$) would be:

$$(f''(x))^2 = c(1 + f'(x)^2)$$

This is not the second derivative of a parabola, hence let us now investigate the curve and find out its equation.
A catenary is a type of curve that assumes a shape, which minimizes the Gravitational Potential Energy.
When a cable (in this case the LED lights) is left to assume a shape in its state of static equilibrium, according to a certain theory in physics, gravitational potential energy is minimized.

[1] Calculus: Project 13. Retrieved October 13, 2015, from http://homepage.math.uiowa.edu/~stroyan/CTLC3rdEd/ProjectsOldCD/estroyan/cd/13/

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With this information, I shall derive the equation of the curve step by step.
The Following figure represents the real life Scenario graphically:

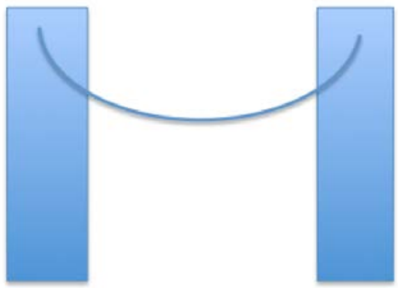


Figure 4

In order to derive the equation, it is essential to derive the equation for the length of a curve, which is determined by the integral of **ds** (distance of a small lengths of the curve).

According to the following figure (Using Pythagoras Theorem I shall calculate "ds" since it is one side of the right angled triangle, other sides being **dx** and **dy**):

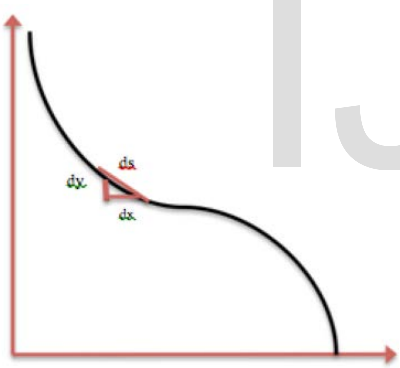


Figure 5

$$\text{ds} = \sqrt{(dx)^2 + (dy)^2}$$

$$\text{ds} = \sqrt{(dx)^2(1 + \left(\frac{dy}{dx}\right)^2)}$$

$$\int \text{ds} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$\text{S} = \int_{x1}^{x2} \sqrt{1 + (y')^2} dx$$

Where y' = dy/dx

This equation is applicable for any normal or irregular curve.
Now, according to the Laws of Physics We know that:

Gravitational Potential energy (P): P = mgh

Where m is the mass of the body, g is acceleration due to free fall and h is the height.

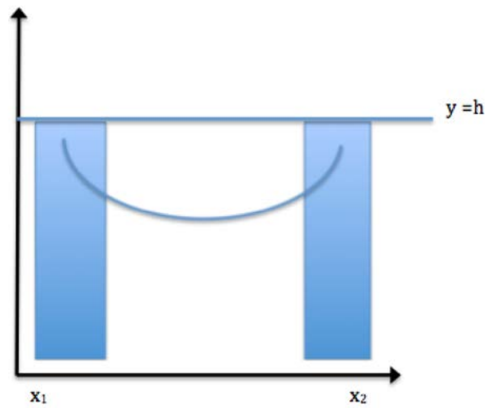


Figure 6 :Drawing Figure 4 on aCartesian plane

It is evident from Figure 6 that **y** is constant because the height is same for both parts of the wall. Applying this in the Potential Energy Equation.

$$P = mgy - \boldsymbol{Eq\ 1}$$

Where m = mass, g= acceleration due to gravity and y is the height of the wall.
Let us consider that the curve is made up of infinitesimal length of "**ds**" and the mass of each of that length is $\boldsymbol{\rho} \times (\boldsymbol{ds})$. Where ρ represents density of the wire.

Rewriting the Potential Energy Equation,

$$\text{P} = \text{mg}y = \rho \times ds \times g \times y - \boldsymbol{Eq\ 2}$$

From Equation 1 and 2

$$\text{P} = \rho g y\sqrt{1 + y'^2}$$

Where, y' = $\frac{dy}{dx}$

Since it has been considered that the cable is of uniform Mass

Total Potential Energy = Integral of Potential energy of small infinitesimal parts of which the curve consists of,

$$\text{P} = \int_a^b \rho \times g \times y\sqrt{1 + y'^2} dx$$

$$= \rho \times g \int_a^b y\sqrt{1 + y'^2}\, dx$$

Where **a** and **b** are the starting and ending points on the curve.
This is the integral for total potential energy, since the curve is said to be minimizing potential energy, I shall find "**y**" which minimizes the integral.

$$\mathrm{F}(\mathrm{y},\mathrm{y}') = y\sqrt{1+y'^2} - \boldsymbol{Eq\ 3}$$

To find out "**y**", which minimizes the Potential Energy, a Derivation of the Euler –**Lagrange Equation** is used. Euler-Lagrange equation states that:
If **A** is defined by the integral=

$$\boldsymbol{A} = \int \boldsymbol{F}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{y}')\boldsymbol{dx}$$

Where $y' = \frac{dy}{dt}$ then **A** has a stationary value when

$$^{2}\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}} - \frac{\boldsymbol{dy}}{\boldsymbol{dx}}\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}'}\right) = \boldsymbol{0}$$

is satisfied.
Euler Lagrange Equation: [3]It is a second order partial differential equation, which describes a function that describes the stationary point of a functional. Using a derivation of the Euler – Lagrange equation, also known as the Beltrami Identity, **y** can be derived,

The Beltrami Identity States:

$$^{4}\boldsymbol{F} - \boldsymbol{y}'\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}'} = \mathbf{C} - \mathbf{Eq\ 4}$$

Substituting Equation 3 in Equation 4:

$$\left(y\sqrt{1+y'^2}\right) - y'\frac{\partial}{\partial y'}\left(y\sqrt{1+y'^2}\right) = Constant$$

$$\left(y\sqrt{1+y'^2}\right) - \frac{yy'^2}{\sqrt{1+y'^2}} = \text{Constant}$$

Multiplying the whole equation by $\sqrt{1+y'^2}$,

$$y\left(1+y'^2\right) - yy'^2 = C \times \sqrt{1+y'^2}$$

Expanding the Equation:

$$y + yy'^2 - yy'^2 = C(\sqrt{1+y'^2})$$

$$y = C(\sqrt{1+y'^2})$$

[2]Euler-Lagrange Differential Equation. Retrieved November 10, 2015, from http://mathworld.wolfram.com/Euler-LagrangeDifferentialEquation.html
[3]http://www.yourdictionary.com/euler-lagrange-equation
[4]Beltrami Identity. Retrieved December 13, 2015, from http://mathworld.wolfram.com/BeltramiIdentity.html

Squaring both sides of the equation

$$y^2 = C^2(1+y'^2)$$

$$\frac{y^2}{C^2} = 1 + y'^2$$

Rearranging the equation

$$\left(\frac{y}{C}\right)^2 - (y')^2 = 1 - \mathbf{Eq\ 5}$$

Deriving the identity (Since Equation 5 looks like the identity written below):

$$(\cosh\theta)^2 - (\sinh\theta)^2 = 1$$

If the curve is a catenary curve the solution should be y = $\mathrm{C} \times \cosh(bx)$

Where cosh (x) is cosine hyperbolic function =

$$\left(\frac{e^x + e^{-x}}{2}\right)$$

Then y' = C b $\sinh(bx)$, where sinh (x) is sine hyperbolic function =

$$\left(\frac{e^x - e^{-x}}{2}\right)$$

$$(\sinh x)^2 = \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} - e^{-2x} - 2}{4}$$

$$(\cosh x)^2 = \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \left(\frac{e^{2x} + e^{-2x} + 2}{4}\right)$$

Subtracting $(\sinh x)^2$ from $(\cosh x)^2$

$$(\cosh x)^2 - (\sinh x)^2 = 1$$

$$\frac{(e^{2x} + e^{-2x} + 2) - (e^{2x} + e^{-2x} - 2)}{4}$$

$$\frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{2}$$

$$\frac{4}{4}$$

$$= 1$$

Hence from the above derivation it has been proved that $(\cosh\theta)^2 - (\sinh\theta)^2 = 1$

Substituting $y = C \times \cosh(bx)$ and its derivative $y' = C \times b \times \sinh(bx)$ in Equation 5

$$\left(\frac{C \times \cosh(bx)}{C}\right)^2 - (C \times b \times \sinh(bx))^2 = 1$$

[5] $$(\cosh(bx))^2 - (C^2 \times b^2 (\sinh bx)^2) = 1$$

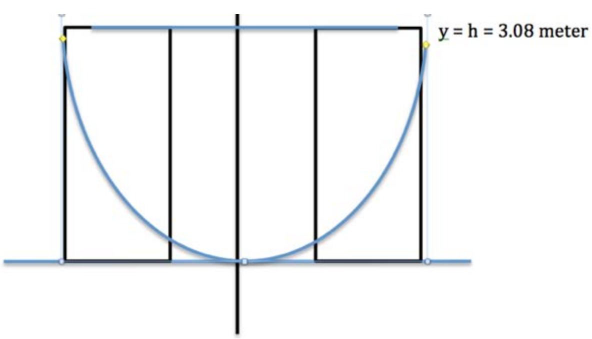
Comparing this Equation with that of Equation of Catenary we get b =1/C

$$F(x) = C \times \cosh\left(\frac{x}{C}\right)$$

Hence, this is the equation of a catenary, thus proving that the shape of the curve assumed by the LED String Lights is that of a Catenary.

2.2 Part 2: Calculating the minimum distance between the two end points of the Catenary such that the LED String lights just touch the ground

Reconstructing the Diagram, the distance between the two points on the wall at which the cable touches the ground can be solved using the following diagram.



With the help of the above drawn graph, I will be calculating the length 2x- (distance between the two points on the wall) at which the cable just touches the ground. In the equation of a catenary curve, C is the ratio of the horizontal component of the tension force to the weight per unit length of the LED String light.
On account of which, C will be subtracted from the equation:
F (x) = C cosh (x/C) – C

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[5] Catenary Curve. (2014). Retrieved January 01, 2016, from https://conversationofmomentum.wordpress.com/2014/08/10/catenary-curve/

$$F(x) = \left(C \times \cosh\left(\frac{x}{C}\right)\right) - C$$

I shall divide the LED lights in two halves for accuracy in calculation = $\frac{22}{2}$ = 11 meter
Each of the 11-meter-long string lights again divided (To maximize accuracy in calculation) by 2 equal to 5.5 meter (distance from center of curve to one of the endpoints). Using these values, I shall calculate C:
The Formula for Arc length = S =

$$\int_{x1}^{x2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_0^{5.5} \sqrt{\left(1 + \left(\sinh\left(\frac{x}{C}\right)\right)^2\right)}\, dx$$

$$S = C \sinh\left(\frac{x}{C}\right) = 5.5 \quad \boldsymbol{Eq\ 6}$$

$$C \times \cosh\left(\frac{x}{C}\right) - C = 3.08 - \boldsymbol{Eq\ 7}$$

Solving Equation 6 and 7 using $(\cosh\theta)^2 - (\sinh\theta)^2 = 1$

$$\left(\frac{3.08 + C}{C}\right)^2 - \left(\frac{5.5}{C}\right)^2 = 1$$

Graphing it (considering two functions: $y = \left(\frac{3.08+C}{C}\right)^2 - \left(\frac{5.5}{C}\right)^2$ and y= 1) and finding out their point of intersection:

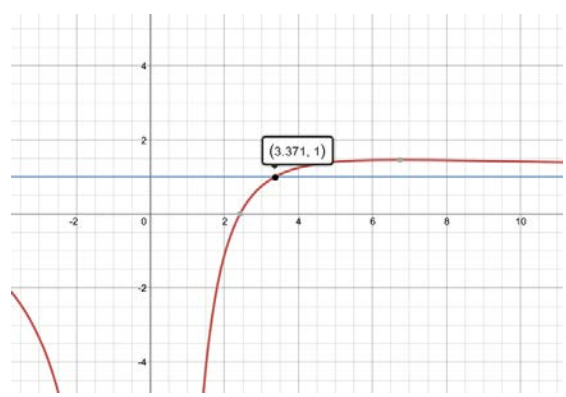

Figure 8 : Graph of : $y = \left(\frac{3.08+C}{C}\right)^2 - \left(\frac{5.5}{C}\right)^2$ and y = 1

By solving it (and from Figure 8) we get C = 3.371

Substituting the value of C in Equation 6 to get:

$$3.371 \times \sinh\left(\frac{x}{3.371}\right) = 5.5 \quad \boldsymbol{Eq\ 8}$$

Graphing **Equation 8** on calculator to get **x = 4.26 meter**:

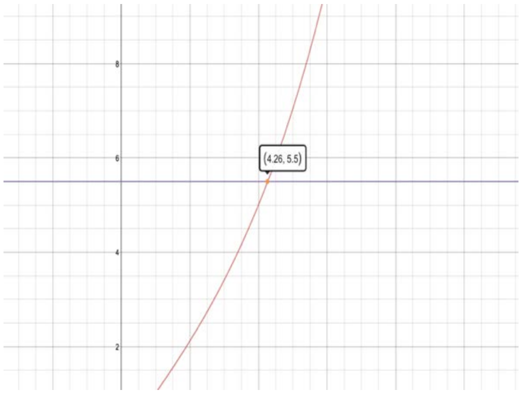

Figure 9

Therefore, the distance between the two end points is:

$$4.26 \times 2 = 8.52$$

Hence the distance between the two end points is **8.52 meter when the LED lights are tangent to the ground**. With this we can conclude that the minimum distance between the two endpoints should be greater than 8.52 meter. If the distance is less than 8.52 meter then the cable will lie on the ground, which is hazardous for people present near and around the LED lights.

The equation of the curve assumed by the LED String light when it just touches the ground is:

$$y = 3.371 \times \cosh\left(\frac{x}{3.371}\right) - 3.371$$

In my analysis, I have concluded with the result that it is crucial to maintain a minimum distance of more than 8.52 meters between the endpoints for safety and security when the value of C =3.371.

2.3 Part 3: Calculating the value of "C" and deducing the length of the curve, which maximizes the area covered by the LED String Lights curve.

To calculate the value of C for which the curve covers maximum area above it, I will try out a range of values for C. I have opted to use this method since there are several conditions that I must satisfy while maximizing the area covered by the curve.
Two of the important conditions are

1. The height of the wall is fixed to 3.08 meter
2. The lowest point of the curve should be above 6 feet (1.8 meter) (The reason I will be explaining later on)

We know that the maximum length of the LED String light is 11 meter (Total distance between two end points). We also know that the minimum distance between the center of the curve to one end point is 4.26 meter. Hence, if the catenary is approaching a straight line the value of **x** in the equation (of total length of the curve):

$$C \times \sinh\left(\frac{x}{C}\right) = 5.5$$

should be greater than 4.26. However, it must be kept in mind that the total length of the curve must be 5.5 meter.

The following table represents different values of "C" for the values of x ranging from 4.26 to 5.5 for my investigation.

S.no.	C	x	Total Area covered by the curve (under it) assumed by the LED String lights
1.	3.454	4.30	9
2.	3.72	4.40	11.256
3.	4.021	4.50	13.88
4.	4.365	4.60	17
5.	4.765	4.70	20.72
6.	5.24	4.80	25.278
7.	5.819	4.90	30.972
8.	6.55	5.00	38.388
9.	7.52	5.10	48.336
10.	8.914	5.20	62.994
11.	9.891	5.25	73.4
12.	11.201	5.30	86.758
13.	16.245	5.40	142.288
14.	18.253	5.42	164.24

Formula to calculate the area covered by the curve (Taking into consideration the equation of the curve when the string lights just touch the ground):

$$\int_0^x \mathrm{C} \times \cosh\left(\frac{x}{C}\right) - 3.371\, dx$$

Therefore, the value for which the curve covers minimum area under the curve leads us to deduce that it covers maximum area above it. But before concluding from the analysis made above, it should be kept in mind that the safety of people at home must be considered while decorating the house. It is of utmost significance to take into consideration the height of the catenary from the ground. An average human height for male is 6 feet (1.8 meter) and female is 5 feet (approx. 1.5 meter). Hence I shall deduce that value of "C" from my analysis of the area under the curve for which the catenary is 6 feet from the ground. (For the

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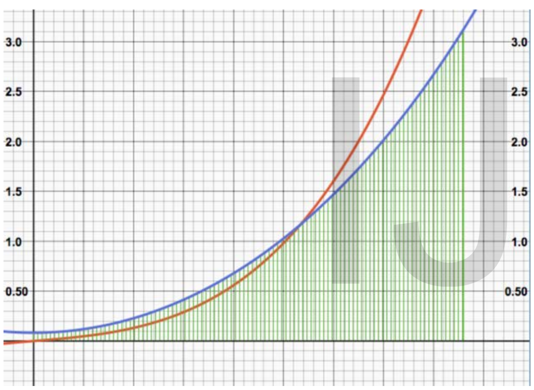
analysis it has been assumed that x-axis represents the ground.)

(Each unit in x-axis and y-axis represents 1 meter)
Note that I am calculating the area under the graph from x =0 to x= (a value depending upon the value of C) because my intention is to find the distance between the endpoints for which the height of the LED String light from the ground is more than 1.8 meter. However, I have found out the total area under the curve by multiplying the result of area under the curve from x = 0 to x = (a value depending on C) into 2. Now I shall give an example of my calculation to illustrate and support my argument.

For example, I start with the value of C = 3.454, Hence x = 4.30. First I calculated the integral myself and then I graph this using software online (http://www.integral-calculator.com)

Graph of

$$\int_0^{4.30} 3.454 \times \cosh\left(\frac{x}{3.454}\right) - 3.371\, dx$$



Calculating half of the area covered under the graph:

$$\int_0^{4.30} 3.454 \times \cosh\left(\frac{x}{3.454}\right) - 3.371\, dx \approx \;\; 4.501 m^2$$

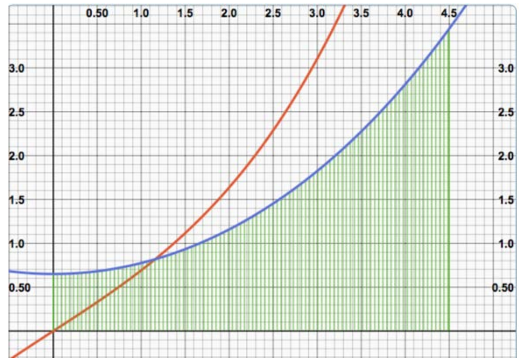
Therefore total area covered under the graph = 2 $\times$ $4.501 = 9\; m^2$
But the lowest point of the curve is around 0.01 meter from the ground, hence it does not satisfy the first condition.

By using similar calculation for the other values of C I shall deduce the optimal value of C.

The following graphs represent the area under the curve and the lowest point of the curve for different values of C.

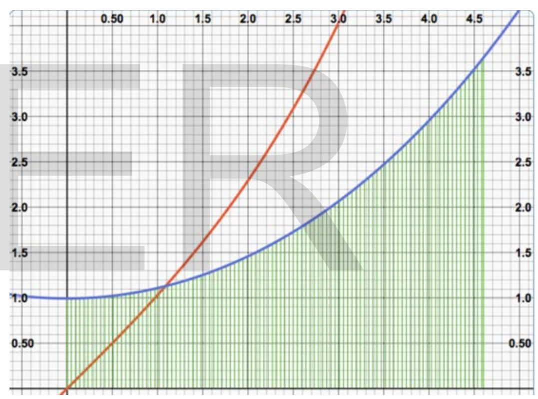
Now let us consider a higher value from the table

For C = 4.021



Area under the Graph = $13.88\; m^2$
In this case the lowest point of the graph is around 0.6 meter from the ground, therefore, I shall take higher values of C and investigate further.

For C = 4.365



Area under the graph (From the Table): 17 m^2
The lowest point in this graph is close to the required condition (1.8 meter) but not greater that 1.8 meter.

For C = 4.765

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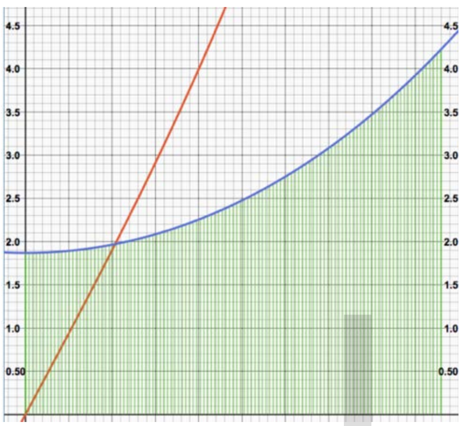
Area Under The graph: $20.72m^2$
From the graph it is evident that the lowest point is around 1.5 meter, hence I shall take a greater value of C.

For C = 5.24



This value of C finally satisfies the condition required. The lowest point is approximately 1.879 meters from the ground.

Total Area under the Graph: $25.278\ m^2$

From this result, we can conclude that the minimum value of "C" for which the area is maximized is 5.24; this satisfies one of our conditions. In this case for x=4.80, y =4.225 meter, but The height of the wall is 3.08 meter, therefore let us first find **x** and then find the length of the LED String light for that height to deduce the distance between the two endpoints for which the curves covers maximum area.

The value of "x", that is the distance from the center of the curve to one of the end points when the height of the wall (3.08 meter) is fixed is 3.48 meters. (Represented in the graph below).

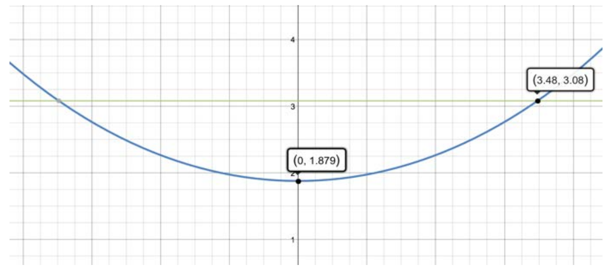


Figure10: Representing the value of x for which y= 3.08 (The height of the wall)

For this value of "x", the length of the cable from x = 0 to x = 3.48 is deduced from the Equation:

$$C \times \sinh\left(\frac{x}{C}\right) = Total\ length\ of\ the\ curve$$

Substituting the values:

$$5.24 \times \sinh\left(\frac{3.48}{5.24}\right) = \ 3.741$$

From the analysis of the situation through modeling of curves and solving equations, it can be concluded that the total length of the wire maximizing the area above the curve is $2 \times 3.741 = 7.482$. The original length of the LED String Light is 22 meters, this solution gives a cost effective way of decorating maximum area of the house, hence fulfilling my last objective of the exploration.

It must be noted that the minimum distance between the center of the curve to one end point of the curve is 4.26 When C =3.371. Since C here is equal to 5.24, the distance between the center of the curve to one of the end point is 3.48 meter, when the lowest point of the curve is 1.897 meter above the ground. In addition, the length of the curve is 3.741 meter, which is less than 5.5 meter. Hence supporting my conclusions made before.

Through my exploration I have been able to solve a crucial issue faced by a common man in any part of the world while celebrating festivals. I have been able to explore the versatile nature of Mathematics and aim to apply this in solving large scope problems in the future.

3 Conclusion and Limitations

On the basis of my investigation, I conclude that the LED String Lights hung on the wall assume the shape of a catenary with the equation: $C \times \cosh\left(\frac{x}{C}\right)$ where C is a constant that depends upon the distance between the two endpoints of the catenary.

The result of the investigation lucidly satisfies the main aim stated in the introduction of the text, however there are several limitations to my exploration that may limit certain aspects of its application. It deals with a limited quantity of data; there is a fair possibility of getting a different result if a different situation with the Catenary is considered. As evident the range within which the exploration has been carried out, it can be implied that derivation of various results and conclusion may not affect large scope problems like it does to the real life situation dealt in this exploration.

4 Further Scope of Application

Applying Calculus of Variations to solve real life problems allowed me to explore application of mathematics in real life crucial situations, using knowledge in relation to achieving a target and moreover the exploration has broadened my knowledge of mathematics. In the future I am eager to explore the concept in detail to unravel solutions to pressing issues present in myriad societies, communities etc. It was very fascinating for me to solve different kinds of equations in deriving the equation of the catenary as well as realizing the affect of the change in a single constant. Nevertheless, learning a new topic in mathematics has ingrained a new curiosity within me for further research. Moreover, I realized Calculus of Variations could be applied in situations of deriving equations of irregular curve, minimizing surface area of revolution. During my research on Calculus of Variations, I came across various other applications of this topic, which I hope to apply in real life. Moreover, this exploration allowed me to appreciate the beauty of mathematics in solving complex problem in a simple manner.

5 References

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